

Irrational Behavior of Bilateral Mock Theta Functions

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Abstract

Ramanujan was the first who introduced the mock theta functions in his last letter to G. H. Hardy in January 1920. Shukla and Ahmad obtained bilateral mock theta functions of order $7, 9, \dots, 2r + 1$, and they have shown that these functions meet the mock theta function's distinctive attribute. This article shows the irrational behavior of bilateral mock theta functions of order $2r + 1$ at the points $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \dots$. Based on this behavior, irrational behavior of bilateral mock theta functions of order $5, 7, 9, \dots$ has also been discussed.

Keywords

Bilateral Mock Theta Function; Cantor Series

1. Introduction

The theory of mock theta functions started from the last letter of Ramanujan to G.H. Hardy in January 1920, which was written from a hospital bed three months before his demise. In the course of it, he said

“I discovered very interesting functions recently which I call ‘Mock Theta functions, \dots ”.

He just provided a list of seventeen instances and a qualitative explanation of the important quality he had observed, without providing a definition of mock theta functions. These seventeen instances were designated as mock theta functions of orders three, five, and seven by Ramanujan. As Hardy commented,

Mock theta function is a function defined by a q -series, convergent when $|q| < 1$ for which we can calculate asymptotic formulae, when q tends to a rational point, $e^{\frac{2\pi ir}{s}}$ of the unit circle of the same degree of precision as those furnished for the ordinary theta functions by the theory of linear transformations.

Watson [16, 17] studied in great detail the mock theta functions of order three and five and introduced three more functions of order three to this collection. The qualitative descriptions given by Ramanujan and subsequent remarks given by various mathematicians from time to time indicate that:

Mock theta function is a function defined by a q -series, convergent for $|q| < 1$ which satisfies the following two conditions:

1. For every root of unity ξ , there is a theta function $\theta_\xi(q)$ such that the difference $f(q) - \theta_\xi(q)$ is bounded as $q \rightarrow \xi$ radially,
2. There is no single θ function which works for all ξ , i.e. there is some root of unity ξ for which $f(q) - \theta_\xi(q)$ is unbounded as $q \rightarrow \xi$ radially.

Mingarelli [6] showed that all Ramanujan’s mock theta functions of order three, Watson’s three additional mock theta functions of order three, the Rogers-Ramanujan q -series, and six mock theta functions of order five take on irrational values at the points $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \dots$.

The *complete* or *bilateral mock theta function* corresponding to a mock theta function like

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q)_n}$$

is defined as

$$f_{1C}(q) = \sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)}}{(-q)_n}.$$

In 1999, Srivastava [15] obtained the eight bilateral mock theta functions of order ‘five’ by using the transformation of Bailey ${}_2\Psi_2$ series and also given some alternative forms of these functions by using the the general transformation of Gasper and Rahman [4] [page - 129 (5.4.3)] for $r = 2$.

Shukla and Ahmad [1, 8–10] obtained bilateral mock theta functions of order $7, 9, \dots, 2r + 1$ by using the transformation (1.1.3) due to Slater [14] for $r = 3, 4, \dots, 2r + 1$ respectively. They have shown that these functions are the limiting cases of the basic hypergeometric series ${}_4\Phi_3, {}_5\Phi_4, \dots, {}_{r+1}\Phi_r$, respectively and proved that they satisfy the characteristic property of the mock theta function. The bilateral mock theta functions of order ‘ $2r+1$ ’ are

$$f_{0C_r}(q) = \sum_{-\infty}^{\infty} \frac{(-1)^{rn} q^{r \frac{(n^2-n)}{2}} q^n}{(-q; q)_n}, \tag{1}$$

$$f_{1C_r}(q) = \sum_{-\infty}^{\infty} \frac{(-1)^{rn} q^{r \frac{(n^2-n)}{2}} q^{2n}}{(-q; q)_n}, \tag{2}$$

$$F_{0C_r}(q) = \sum_{-\infty}^{\infty} \frac{(-1)^{rn} q^{r(n^2-n)} q^{2n}}{(q; q^2)_n}, \tag{3}$$

$$F_{1C_r}(q^2) = \sum_{-\infty}^{\infty} \frac{(-1)^{rn} q^{r(2n^2-2n)} q^{8n}}{(q^6; q^4)_n}, \tag{4}$$

$$\Phi_{0C_r}(q) = \sum_{-\infty}^{\infty} \frac{(-1)^{rn} q^{rn^2}}{(-q; q^2)_n}, \tag{5}$$

$$\Phi_1 c_r(q) = \sum_{-\infty}^{\infty} (-1)^{rn} q^{(r-1)(n^2+2n)} (-q; q^2)_n, \tag{6}$$

$$\Psi_0 c_r(q) = \sum_{-\infty}^{\infty} (-1)^{rn} q^{(r-1)\frac{(n^2+3n)}{2}} (-q; q)_n, \tag{7}$$

$$\Psi_1 c_r(q) = \sum_{-\infty}^{\infty} \frac{(-1)^{r(n+1)} q^{\frac{rn(n+1)}{2}}}{2(-q; q)_n}. \tag{8}$$

Singh [11, 12] showed that bilateral mock theta functions of order 5 and 9 are irrational at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \dots$. Recently, Singh [13] showed that four of the bilateral mock theta functions of order 7 are irrational at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \dots$, and four are irrational at $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$. In this paper, the irrational behavior of bilateral mock theta functions of order $2r + 1$ at an infinite number of points has been discussed, and the discussion includes the theorems of Oppenheim [7] regarding the Cantor series, which are stated in the preliminaries. In Section 3, two theorems have been proved on the irrational behavior of bilateral mock theta functions of order $2r + 1$, while the theorems on the irrational behavior of bilateral mock theta functions of order 5, 7, 9, \dots have been deduced in Section 4.

2. Preliminaries

The following q -notations and some standard results have been used:

For $|q| < 1$,

$$(a)_n = (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad (a; q)_0 = 1$$

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n.$$

A generalised basic hypergeometric series with base q is defined as

$${}_r\Phi_{r-1} \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_{r-1} \end{matrix}; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(q; q)_n (b_1, \dots, b_{r-1}; q)_n} z^n, \quad |z| < 1.$$

A bilateral basic hypergeometric series with base q is defined as

$${}_r\Psi_r \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix}; q, z \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_r; q)_n} z^n \quad \text{for } \left| \frac{b_1 \cdots b_r}{a_1 \cdots a_r} \right| < |z| < 1,$$

and

$$(a; q^k)_{-n} = \frac{(-q^k/a)^n}{(q^k/a; q^k)_n} q^{\frac{kn(n-1)}{2}}.$$

Throughout this paper, we use the following form of the Cantor (infinite) series and two theorems given by Oppenheim regarding this series:

2.1 The Cantor Series and the Theorems of Oppenheim

Cantor [2] provided a criterion for determining the irrationality of a real number σ given by the infinite series of the form

$$\sigma = \sum_{n=1}^{\infty} \frac{j_n}{k_1 k_2 \cdots k_n} \tag{9}$$

where the j_i ($i \geq 0$), k_i ($i \geq 1$) are integers to have irrational sums with the conditions $k_i \geq 2$, $0 \leq j_i \leq k_i - 1$ and for every integer $m \geq 1$ there is an n such that $m \mid k_1 k_2 \cdots k_n$. Cantor showed that σ is irrational if and only if the $j_i > 0$ infinitely often and $j_i < k_i - 1$ infinitely often.

Oppenheim [7] and Diananda & Oppenheim [3] extended the above criterion to the case where the j_i can have both signs and dropped the divisibility condition on the product of the first n k -s. Further development in the theory came in a paper by Hancl and Tijdeman [5] in which the use of Cantor-Oppenheim a priori condition $j_i \leq k_i - 1$ is avoided.

Theorem 2.1 (Oppenheim [7], Theorem 4) *Let (j_n) , (k_n) be two sequences of integers with $k_n \geq 2$, $0 \leq j_n \leq k_n - 1$. If $j_n > 0$ infinitely often and if there is a subsequence i_n such that $k_{i_n} \rightarrow \infty$ and $j_{i_n}/k_{i_n} \rightarrow 0$ as $n \rightarrow \infty$, then σ as defined in (9) is irrational.*

Theorem 2.2 (Oppenheim [7], Theorem 8) *Let (j_n) , (k_n) be two sequences of integers with $k_n \geq 2$, $|j_n| \leq k_n - 1$. Furthermore, let $j_m j_n < 0$ for some $m > i$, $n > i$ for any assigned integer i . If there is a subsequence i_n such that $k_{i_n} \rightarrow \infty$ and $j_{i_n}/k_{i_n} \rightarrow 0$ as $n \rightarrow \infty$, then σ given by (9) is irrational.*

3. Irrationality of Bilateral Mock Theta Functions of order $2r + 1$

The following two theorems on the irrationality of bilateral mock theta functions of order $2r + 1$ have been proved in Subsections 3.1 and 3.2:

Theorem 3.1 *The bilateral mock-theta functions $F_0 c_r(q)$, $F_1 c_r(q^2)$, $\Phi_0 c_r(q)$, and $\Phi_1 c_r(q)$ take on irrational values at $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$.*

Theorem 3.2 *The bilateral mock theta functions $f_0 c_r(q)$, $f_1 c_r(q)$, $\Psi_0 c_r(q)$, and $\Psi_1 c_r(q)$ take on irrational values at*

(i) $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ when $r = 3, 5, 7, \dots$, and

(ii) $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$ when $r = 2, 4, 6, \dots$.

3.1 Proof of Theorem 3.1

In this section, the proofs of the irrationality of functions $F_0 c_r(q)$, $F_1 c_r(q^2)$, $\Phi_0 c_r(q)$, and $\Phi_1 c_r(q)$ are given separately one by one:

3.1.1 $F_0 c_r(q)$ is irrational at $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$

The function $F_0 c_r(q)$ can be written as

$$F_0 c_r(q) = 1 + S_{F_0}(q) + \sum_{n=1}^{\infty} (-1)^{(r-1)n} q^{(r-1)n^2 + (r-2)n} (q; q^2)_n. \tag{10}$$

It follows that $F_0 c_r(q)$ is irrational if and only if $S_{F_0}(q)$ is irrational, and

$$S_{F_0}(q) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} q^{rn^2 - (r-2)n}}{(1-q)(1-q^3) \cdots (1-q^{2n-1})} \tag{11}$$

Let $p, q \in C, q \neq 0$, then

$$S_{F_0} \left(\frac{p}{q} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} p^{rn^2-(r-2)n} q^{(r-2)n}}{q^{(r-1)n^2} (q-p) (q^3-p^3) \dots (q^{2n-1}-p^{2n-1})} \tag{12}$$

Taking $p = 1$ in preceding expression, we get

$$S_{F_0} \left(\frac{1}{q} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} q^{(r-2)n}}{q^{(r-1)n} (q-1) q^{3(r-1)n} (q^3-1) \dots q^{(2n-1)(r-1)n} (q^{2n-1}-1)} \tag{13}$$

Since for every $q \geq 2, q \in Z$, the above expression is a Cantor series given by (9) with the identifications $j_n = (-1)^{rn} q^{(r-2)n}, k_n = q^{(2n-1)(r-1)} (q^{2n-1}-1)$. For every $q \geq 2$ and an integer $n \geq 1, k_n \geq 2, k_n - 1 > |j_n| > 0, k_n \rightarrow \infty, j_n/k_n \rightarrow 0$ as $n \rightarrow \infty$. Hence, all the conditions of Theorem 2.1 are satisfied when $r = 2, 4, 6, \dots$ and all the conditions of Theorem 2.2 are satisfied when $r = 3, 5, 7, \dots$, therefore $S_{F_0} \left(\frac{1}{q} \right)$ is irrational for every integer $q \geq 2$.

Again, taking $p = -1$ in (12), we obtain

$$S_{F_0} \left(-\frac{1}{q} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} q^{(r-2)n}}{q^{(r-1)n} (q+1) q^{3(r-1)n} (q^3+1) \dots q^{(2n-1)(r-1)n} (q^{2n-1}+1)} \tag{14}$$

Hence $S_{F_0} \left(-\frac{1}{q} \right)$ is a Cantor series as given by (9) with the identifications $j_n = (-1)^{rn} q^{(r-2)n}, k_n = q^{(2n-1)(r-1)} (q^{2n-1}+1)$. For every integer $q \geq 2$ and any $n \geq 1, k_n \geq 2, k_n - 1 > |j_n| > 0, k_n \rightarrow \infty$ & $j_n/k_n \rightarrow 0$ as $n \rightarrow \infty$. Hence, by Theorem 2.1, $S_{F_0} \left(-\frac{1}{q} \right)$ is irrational for every integer $q \geq 2$ when $r = 2, 4, 6, \dots$, and by Theorem 2.2, $S_{F_0} \left(-\frac{1}{q} \right)$ is irrational for every integer $q \geq 2$ when $r = 3, 5, 7, \dots$.

Since $S_{F_0} \left(\frac{1}{q} \right)$ and $S_{F_0} \left(-\frac{1}{q} \right)$ are irrational for every integer $q \geq 2$, so $S_{F_0}(q)$ is irrational for $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$.

3.1.2 $F_{1c_r}(q^2)$ is irrational at $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$

The function $F_{1c_r}(q^2)$ can be written as

$$F_{1c_r}(q^2) = 1 + S_{F_1}(q) + \sum_{n=1}^{\infty} (-1)^{(r-1)n} q^{2(r-1)n^2+2(r-2)n} (q^{-2}; q^4)_n \tag{15}$$

It is noted that $F_{1c_r}(q^2)$ is irrational if and only if $S_{F_1}(q)$ is irrational, and

$$S_{F_1}(q) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} q^{2rn^2-2(r-4)n}}{(1-q^6)(1-q^{10}) \dots (1-q^{4n+2})} \tag{16}$$

As before, for $p, q \in C, q \neq 0$, we have

$$S_{F_1} \left(\frac{p}{q} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} p^{2rn^2-2(r-4)n} q^{2(r-2)n}}{q^{2(r-1)n^2} (q^6-p^6) (q^{10}-p^{10}) \dots (q^{4n+2}-p^{4n+2})}. \tag{17}$$

Substituting $p = \pm 1$ in (17), we obtain

$$S_{F_1} \left(\pm \frac{1}{q} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} q^{2(r-2)n}}{q^{2(r-1)n} (q^6-1) q^{6(r-1)n} (q^{10}-1) \dots q^{2(2n-1)(r-1)n} (q^{4n+2}-1)} \tag{18}$$

The sum above is a Cantor series as given by (9) with the identifications $j_n = (-1)^{rn} q^{2(r-2)n}$, $k_n = q^{2(2n-1)(r-1)} (q^{4n+2} - 1)$. Also, for every integer $q \geq 2$, and any $n \geq 1$, $|j_n| > 0$, $k_n \geq 2$, $k_n - 1 > |j_n| > 0$, $k_n \rightarrow \infty$ & $j_n/k_n \rightarrow 0$ as $n \rightarrow \infty$. Hence, all the conditions of Theorem 2.1 are satisfied when $r = 2, 4, 6, \dots$, and all the conditions of Theorem 2.2 are satisfied when $r = 3, 5, 7, \dots$ for $q \geq 2$, so $S_{F_1} \left(\pm \frac{1}{q}\right)$ is irrational for every integer $q \geq 2$. Thus, $S_{F_1}(q)$ is irrational for $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$.

3.1.3 $\Phi_{0C_r}(q)$ is irrational at $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$

In this case, we write the function $\Phi_{0C_r}(q)$ as

$$\Phi_{0C_r}(q) = 1 + S_{\Phi_0}(q) + \sum_{n=1}^{\infty} (-1)^{rn} q^{(r-1)n^2} (-q; q^2)_n \tag{19}$$

It follows that $\Phi_{0C_r}(q)$ is irrational if and only if $S_{\Phi_0}(q)$ is irrational, and

$$S_{\Phi_0}(q) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} q^{rn^2}}{(1+q)(1+q^3)\dots(1+q^{2n-1})} \tag{20}$$

If $p, q \in C$, $q \neq 0$, we see that

$$S_{\Phi_0}\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} p^{rn^2}}{q^{(r-1)n^2} (q+p)(q^3+p^3)\dots(q^{2n-1}+p^{2n-1})} \tag{21}$$

Now, setting $p = 1$ in (21), we obtain

$$S_{\Phi_0}\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{rn}}{q^{(r-1)} (q+1) q^{3(r-1)} (q^3+1)\dots q^{(2n-1)(r-1)} (q^{2n-1}+1)} \tag{22}$$

It is observed that the above sum is a Cantor series as given by (9) with the identifications $j_n = (-1)^{rn}$, $k_n = q^{(2n-1)(r-1)} (q^{2n-1} + 1)$ for every n . For every integer $q \geq 2$, and any $n \geq 1$, all the conditions of Theorem 2.1 are satisfied when $r = 2, 4, 6, \dots$, and all the conditions of Theorem 2.2 are satisfied when $r = 3, 5, 7, \dots$, so $S_{\Phi_0}\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Set again $p = -1$ in (21), we get

$$S_{\Phi_0}\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q^{(r-1)} (q-1) q^{3(r-1)} (q^3-1)\dots q^{(2n-1)(r-1)} (q^{2n-1}-1)} \tag{23}$$

The sum above $S_{\Phi_0}\left(-\frac{1}{q}\right)$ is a Cantor series as given by (9) with $j_n = 1$, $k_n = q^{(2n-1)(r-1)} (q^{2n-1} - 1)$ for every n . For every integer $q \geq 2$, and any $n \geq 1$, being all the conditions of Theorem 2.1 satisfied, $S_{\Phi_0}\left(-\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

As we have seen that $S_{\Phi_0}\left(\frac{1}{q}\right)$ and $S_{\Phi_0}\left(-\frac{1}{q}\right)$ are irrational for every integer $q \geq 2$, therefore $S_{\Phi_0}(q)$ is irrational for $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$.

3.1.4 $\Phi_{1c_r}(q)$ is irrational at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \dots$

The function $\Phi_{1c_r}(q)$ can be expressed as

$$\Phi_{1c_r}(q) = \sum_{n=0}^{\infty} (-1)^{rn} q^{(r-1)n^2+2n} (-q; q^2)_n + S_{\Phi_1}(q) \tag{24}$$

It is clear that $\Phi_{1c_r}(q)$ is irrational if and only if $S_{\Phi_1}(q)$ is irrational, and

$$S_{\Phi_1}(q) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} q^{rn^2-2(r-1)n}}{(1+q)(1+q^3)\dots(1+q^{2n-1})} \tag{25}$$

Whenever $p, q \in C, q \neq 0$, then

$$S_{\Phi_1}\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} p^{rn^2-2(r-1)n} q^{2(r-1)n}}{q^{(r-1)n^2} (q+p)(q^3+p^3)\dots(q^{2n-1}+p^{2n-1})} \tag{26}$$

Setting $p = 1$ in (26), we obtain

$$S_{\Phi_1}\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} q^{2(r-1)n}}{q^{(r-1)(n^2)} (q+1)q^{3(r-1)} (q^3+1)\dots q^{(2n-1)(r-1)} (q^{2n-1}+1)} \tag{27}$$

The sum $S_{\Phi_1}\left(\frac{1}{q}\right)$ is a Cantor series given by (9) with $k_n = q^{(2n-1)(r-1)} (q^{2n-1} + 1)$, and $j_n = (-1)^{rn} q^{2(r-1)n}$ for every n . Also, for every integer $q \geq 2$, and any $n \geq 1$, all the conditions of Theorem 2.1 are satisfied when $r = 2, 4, 6, \dots$, and all the conditions of Theorem 2.2 are satisfied when $r = 3, 5, 7, \dots$, therefore $\Phi_{0S_r}\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Again, setting $p = -1$ in (26) and simplifying, we get

$$S_{\Phi_1}\left(-\frac{1}{q}\right) = \frac{q^{r-1}}{q-1} + \frac{1}{q^{r-1}(q-1)} \times \sum_{n=1}^{\infty} \frac{q^{2(r-1)(n+1)}}{q^{3(r-1)} (q^3-1) q^{5(r-1)} (q^5-1)\dots q^{(2n+1)(r-1)} (q^{2n+1}-1)} \tag{28}$$

The sum in the R.H.S. of the above expression is a Cantor series given by (9) with $j_n = q^{2(r-1)(n+1)}$, $k_n = q^{(2n+1)(r-1)} (q^{2n+1} - 1)$ for every n . For every integer $q \geq 2$, and any $n \geq 1$, all the conditions of Theorem 2.1 are satisfied, so $S_{\Phi_1}\left(-\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Since $S_{\Phi_1}\left(\frac{1}{q}\right)$ and $S_{\Phi_1}\left(-\frac{1}{q}\right)$ are irrational for every integer $q \geq 2$, therefore $S_{\Phi_1}(q)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \dots$.

3.2 Proof of Theorem 3.2

In this section, the irrational behavior of the functions $f_{0c_r}(q), f_{1c_r}(q), \Psi_{0c_r}(q)$, and $\Psi_{1c_r}(q)$ has been shown separately, one by one :

3.2.1 $f_0c_r(q)$ is irrational at (i) $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ when $r = 3, 5, 7, \dots$,
and (ii) $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \dots$ when $r = 2, 4, 6, \dots$

The function $f_0c_r(q)$ can be written as

$$f_0c_r(q) = 1 + S_{f_0}(q) + \sum_{n=1}^{\infty} (-1)^{rn} q^{\frac{(r-1)n^2+(r-1)n}{2}} (-1; q)_n \tag{29}$$

It follows that $f_0c_r(q)$ is irrational if and only if $S_{f_0}(q)$ is irrational, and

$$S_{f_0}(q) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} q^{\frac{rn^2-(r-2)n}{2}}}{(1+q)(1+q^2)\dots(1+q^n)} \tag{30}$$

For $p, q \in C, q \neq 0$, we have

$$S_{f_0}\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} p^{\frac{rn^2-(r-2)n}{2}} q^{\frac{(r-1)n}{2}}}{q^{\frac{(r-1)n^2}{2}} (q+p)(q^2+p^2)\dots(q^n+p^n)} \tag{31}$$

Setting $p = 1$ in (31), we get

$$S_{f_0}\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} q^{\frac{(r-1)n}{2}}}{q^{\frac{r-1}{2}} (q+1) q^{3\frac{(r-1)}{2}} (q^2+1)\dots q^{\frac{(2n-1)(r-1)}{2}} (q^n+1)} \tag{32}$$

The sum $S_{f_0}\left(\frac{1}{q}\right)$ is a Cantor series given by (9) with the identifications $j_n = (-1)^{rn} q^{\frac{(r-1)n}{2}}$, $k_n = q^{\frac{(2n-1)(r-1)}{2}} (q^n+1)$. For every integer $q \geq 2$, and any $n \geq 1$ all the conditions of Theorem 2.1 are satisfied when $r = 2, 4, 6, \dots$, and all the conditions of Theorem 2.2 are satisfied when $r = 3, 5, 7, \dots$, so $S_{f_0}\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Again, taking $p = -1$ in the sum (31), we obtain

$$S_{f_0}\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{\frac{rn^2+(r+2)n}{2}} q^{\frac{(r-1)n}{2}}}{q^{\frac{(r-1)n^2}{2}} (q-1)(q^2+1)\dots(q^n+(-1)^n)} \tag{33}$$

Case (i): If $r = 2, 4, 6, \dots$, then $(-1)^{\frac{rn^2+(r+2)n}{2}} = (-1)^n$, so

$$S_{f_0}\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{(r-1)n}{2}}}{q^{\frac{r-1}{2}} (q-1) q^{3\frac{(r-1)}{2}} (q^2+1)\dots q^{\frac{(2n-1)(r-1)}{2}} (q^n+(-1)^n)} \tag{34}$$

The sum $S_{f_0}\left(-\frac{1}{q}\right)$ is a Cantor series given by (9) with

$$j_n = (-1)^n q^{\frac{(r-1)n}{2}}, k_n = q^{\frac{(2n-1)(r-1)}{2}} (q^n+(-1)^n).$$

For every integer $q \geq 2$, and any $n \geq 1$, all the conditions of Theorem 2.2 are satisfied, so $S_{f_0}\left(-\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$ when $r = 2, 4, 6, \dots$.

Case (ii): If $r = 3, 5, 7, \dots$, then $(-1)^{\frac{rn^2+(r+2)n}{2}}$ is not fixed for all $n \geq 1$ as $(-1)^n$ or $(-1)^{2n}$. In this case, Theorem 2.1 and Theorem 2.2 cannot be applied.

As $S_{f_0} \left(\frac{1}{q} \right)$ has been shown to be irrational for every integer $q \geq 2$, but $S_{f_0} \left(-\frac{1}{q} \right)$ is irrational when $r = 2, 4, 6, \dots$, so $S_{f_0}(q)$ is irrational at

(i) $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ when $r = 3, 5, 7, \dots$, and

(ii) $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$ when $r = 2, 4, 6, \dots$.

3.2.2 $f_{1c_r}(q)$ is irrational at (i) $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ when $r = 3, 5, 7, \dots$, and (ii) $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$ when $r = 2, 4, 6, \dots$

The function $f_{1c_r}(q)$ can be written as

$$f_{1c_r}(q) = 1 + S_{f_1}(q) + \sum_{n=1}^{\infty} (-1)^{rn} q^{\frac{(r-1)n^2+(r-3)n}{2}} (-1; q)_n \tag{35}$$

where

$$S_{f_1}(q) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} q^{\frac{rn^2-(r-4)n}{2}}}{(1+q)(1+q^2)\dots(1+q^n)} \tag{36}$$

Whenever it is defined, $f_{1c_r}(q) \notin Q$, iff $S_{f_1}(q) \notin Q$. Let $p, q \in C, q \neq 0$, then

$$S_{f_1} \left(\frac{p}{q} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} p^{\frac{rn^2-(r-4)n}{2}} q^{\frac{(r-3)n}{2}}}{q^{\frac{(r-1)n^2}{2}} (q+p)(q^2+p^2)\dots(q^n+p^n)} \tag{37}$$

Setting $p = 1$ in (37), we get

$$S_{f_1} \left(\frac{1}{q} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} q^{\frac{(r-3)n}{2}}}{q^{\frac{r-1}{2}} (q+1) q^{\frac{3(r-1)}{2}} (q^2+1)\dots q^{\frac{(2n-1)(r-1)}{2}} (q^n+1)} \tag{38}$$

The sum $S_{f_1} \left(\frac{1}{q} \right)$ is a Cantor series as given by (9) with the identifications $k_n = q^{\frac{(2n-1)(r-1)}{2}} (q^n+1)$, $j_n = (-1)^{rn} q^{\frac{(r-3)n}{2}}$. For every integer $q \geq 2$ and any $n \geq 1$, $|j_n| > 0$, $k_n \geq 2$, $k_n - 1 > |j_n| > 0$, $k_n \rightarrow \infty$ & $j_n/k_n \rightarrow 0$ as $n \rightarrow \infty$. Hence, by Theorem 2.2, $S_{f_1} \left(\frac{1}{q} \right)$ is irrational for every integer $q \geq 2$ when $r = 3, 5, 7, \dots$, and by Theorem 2.1, $S_{f_1} \left(\frac{1}{q} \right)$ is irrational for every integer $q \geq 2$ when $r = 2, 4, 6, \dots$.

Setting again $p = -1$ in (37), we obtain

$$S_{f_1} \left(-\frac{1}{q} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{\frac{rn^2+(r+4)n}{2}} q^{\frac{(r-3)n}{2}}}{q^{\frac{(r-1)n^2}{2}} (q-1)(q^2+1)\dots(q^n+(-1)^n)} \tag{39}$$

Case (i): If $r = 2, 4, 6, \dots$, then $(-1)^{\frac{rn^2+(r+4)n}{2}} = 1$, so

$$S_{f_1} \left(-\frac{1}{q} \right) = \sum_{n=1}^{\infty} \frac{q^{\frac{(r-3)n}{2}}}{q^{\frac{r-1}{2}} (q-1) q^{\frac{3(r-1)}{2}} (q^2+1)\dots q^{\frac{(2n-1)(r-1)}{2}} (q^n+(-1)^n)} \tag{40}$$

The sum $S_{f_1} \left(-\frac{1}{q} \right)$ is a Cantor series as given by (9) with the identifications $j_n = q^{\frac{(r-3)n}{2}}$ and $k_n = q^{\frac{(2n-1)(r-1)}{2}} (q^n+(-1)^n)$. For every integer $q \geq 2$, and any $n \geq 1$, all the conditions of Theorem 2.1 are satisfied, so $S_{f_1} \left(-\frac{1}{q} \right)$ is irrational for every integer $q \geq 2$ when $r = 2, 4, 6, \dots$.

Case (ii): If $r = 3, 5, 7, \dots$, then $(-1)^{\frac{rn^2+(r+4)n}{2}}$ is not fixed for all $n \geq 1$ as $(-1)^n$ or $(-1)^{2n}$. In this case, Theorem 2.1 and Theorem 2.2 cannot be applied.

Since $S_{f_1}\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$, and $S_{f_1}\left(-\frac{1}{q}\right)$ is irrational when $r = 2, 4, 6, \dots$, so $S_{f_1}(q)$ is irrational at

(i) $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ when $r = 3, 5, 7, \dots$, and

(ii) $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \dots$ when $r = 2, 4, 6, \dots$.

3.2.3 $\Psi_0 c_r(q)$ is irrational at (i) $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ when $r = 3, 5, 7, \dots$, and (ii) $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \dots$ when $r = 2, 4, 6, \dots$

As in the preceding cases, the function $\Psi_0 c_r(q)$ can be written as

$$\Psi_0 c_r(q) = \sum_{n=0}^{\infty} (-1)^{rn} q^{\frac{(r-1)(n^2+3n)}{2}} (-q; q)_n + S_{\Psi_0}(q) \tag{41}$$

where

$$S_{\Psi_0}(q) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} q^{\frac{rn^2-(3r-2)n}{2}}}{(1+1)(1+q)(1+q^2)\dots(1+q^{n-1})} \tag{42}$$

It is clear that $\Psi_0 c_r(q)$ is irrational if and only if $S_{\Psi_0}(q)$ is irrational. For $p, q \in C, q \neq 0$, then

$$S_{\Psi_0}\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{rn} p^{\frac{rn^2-(3r-2)n}{2}} q^{\frac{3(r-1)n}{2}}}{q^{\frac{(r-1)n^2}{2}} (1+1)(q+p)(q^2+p^2)\dots(q^{n-1}+p^{n-1})} \tag{43}$$

Taking $p = 1$ in (43) and simplifying, we get

$$S_{\Psi_0}\left(\frac{1}{q}\right) = \frac{(-1)^r q^{r-1}}{2} + \frac{q^{r-1}}{2(q+1)} + \frac{1}{2(q+1)} \sum_{n=1}^{\infty} \frac{(-1)^{rn} q^{r-1}}{q^{r-1}(q^2+1)q^{2(r-1)}(q^3+1)\dots q^{n(r-1)}(q^{n+1}+1)} \tag{44}$$

It is clear that $S_{\Psi_0}\left(\frac{1}{q}\right)$ is irrational if the sum on the right of (44) is irrational, and the sum on the right of (44) is a Cantor series as given by (2.1) with $k_n = q^{n(r-1)}(q^{n+1}+1)$, $j_n = (-1)^{rn} q^{r-1}$. For every integer $q \geq 2$ and any integer $n \geq 1$, $|j_n| > 0$, $k_n \geq 2$, $k_n - 1 > |j_n| > 0$, $k_n \rightarrow \infty$, $\frac{j_n}{k_n} \rightarrow 0$ as $n \rightarrow \infty$. Hence, by Theorem 2.2, $S_{\Psi_0}\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$ when $r = 3, 5, 7, \dots$, and by Theorem 2.1, $S_{\Psi_0}\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$ when $r = 2, 4, 6, \dots$.

Setting again $p = -1$ in (43), we obtain

$$S_{\Psi_0}\left(-\frac{1}{q}\right) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{\frac{rn^2-(r-2)n}{2}} q^{\frac{3(r-1)n}{2}}}{q^{\frac{(r-1)n^2}{2}} (q-1)(q^2+1)\dots(q^{n-1}+(-1)^{n-1})}. \tag{45}$$

Case (i): If $r = 2, 4, 6, \dots$, then $(-1)^{\frac{rn^2-(r-2)n}{2}} = (-1)^n$, so

$$S_{\Psi_0} \left(-\frac{1}{q} \right) = \frac{-q^{r-1}}{2} + \frac{q^{r-1}}{2(q-1)} + \frac{1}{2(q-1)} \times \sum_{n=1}^{\infty} \frac{(-1)^n q^{r-1}}{q^{r-1} (q^2 + 1) q^{2(r-1)} (q^3 - 1) \dots q^{n(r-1)} (q^{n+1} + (-1)^{n+1})} \tag{46}$$

It is clear that $S_{\Psi_0} \left(-\frac{1}{q} \right)$ is irrational if and only if the sum on the right of (46) is irrational. Comparing the sum on the right of (46) with Cantor series as given by (9), we get $k_n = q^{n(r-1)} (q^{n+1} + (-1)^{n+1})$ and $j_n = (-1)^n q^{r-1}$. For every integer $q \geq 2$ and any $n \geq 1$, all the conditions of Theorem 2.2 are satisfied, so $S_{\Psi_0} \left(-\frac{1}{q} \right)$ is irrational for every integer $q \geq 2$ when $r = 2, 4, 6, \dots$.

Case (ii): If $r = 3, 5, 7, \dots$, then $(-1)^{\frac{rn^2-(r-2)n}{2}}$ is not fixed for all $n \geq 1$ as $(-1)^n$ or $(-1)^{2n}$, so Theorem 2.1 and Theorem 2.2 cannot be applied.

Since $S_{\Psi_0} \left(\frac{1}{q} \right)$ is irrational for every integer $q \geq 2$, and $S_{\Psi_0} \left(-\frac{1}{q} \right)$ is irrational when $r = 2, 4, 6, \dots$, so $S_{\Psi_0}(q)$ is irrational at

(i) $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ when $r = 3, 5, 7, \dots$, and

(ii) $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$ when $r = 2, 4, 6, \dots$.

3.2.4 $\Psi_{1c_r}(q)$ is irrational at (i) $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ when $r = 3, 5, 7, \dots$, and (ii) $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$ when $r = 2, 4, 6, \dots$

We express the function $\Psi_{1c_r}(q)$ as

$$\Psi_{1c_r}(q) = \frac{(-1)^r}{2} + \frac{1}{2} S_{\Psi_1}(q) + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{r(-n+1)} q^{\frac{(r-1)n(n-1)}{2}} (-1; q)_n \tag{47}$$

where

$$S_{\Psi_1}(q) = \sum_{n=1}^{\infty} \frac{(-1)^{r(n+1)} q^{\frac{rn(n+1)}{2}}}{(1+q)(1+q^2)\dots(1+q^n)} \tag{48}$$

Whenever it is defined, $\Psi_{1c_r}(q) \notin Q$ iff $S_{\Psi_1}(q) \notin Q$. For $p, q \in C$ and $q \neq 0$, we have

$$S_{\Psi_1} \left(\frac{p}{q} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{r(n+1)} p^{\frac{rn(n+1)}{2}}}{q^{\frac{(r-1)n(n+1)}{2}} (q+p)(q^2+p^2)\dots(q^n+p^n)} \tag{49}$$

Setting $p = 1$ in (49), we get

$$S_{\Psi_1} \left(\frac{1}{q} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{r(n+1)}}{q^{r-1} (q+1) q^{2(r-1)} (q^2+1) \dots q^{n(r-1)} (q^n+1)} \tag{50}$$

The sum $S_{\Psi_1} \left(\frac{1}{q} \right)$ is a Cantor series as given by (2.1) with the identifications $k_n = q^{n(r-1)} (q^n + 1)$, $j_n = (-1)^{r(n+1)}$ for all n . For every integer $q \geq 2$ and any $n \geq 1$, all the conditions of Theorem 2.1 are satisfied when $r = 2, 4, 6, \dots$, and all the conditions of Theorem 2.2 are satisfied when $r = 3, 5, 7, \dots$, therefore $S_{\Psi_1} \left(\frac{1}{q} \right)$ is irrational for every integer $q \geq 2$.

Setting again $p = -1$ in (49), we get

$$S_{\Psi_1} \left(-\frac{1}{q} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{r \frac{(n^2+3n+2)}{2}}}{q^{r-1} (q-1) q^{2(r-1)} (q^2+1) \cdots q^{n(r-1)} (q^n + (-1)^n)} \quad (51)$$

Case (i): If $r = 2, 4, 6, \dots$, then $(-1)^{\frac{r(n+1)(n+2)}{2}} = 1$, so

$$S_{\Psi_1} \left(-\frac{1}{q} \right) = \sum_{n=1}^{\infty} \frac{1}{q^{r-1} (q-1) q^{2(r-1)} (q^2+1) \cdots q^{n(r-1)} (q^n + (-1)^n)} \quad (52)$$

The above sum is a Cantor series as given by (9) with the identifications $j_n = 1$ and $k_n = q^{n(r-1)} (q^n + (-1)^n)$. For every integer $q \geq 2$ and any $n \geq 1$, all the conditions of Theorem 2.1 are satisfied, so $S_{\Psi_1} \left(-\frac{1}{q} \right)$ is irrational for every integer $q \geq 2$ when $r = 2, 4, 6, \dots$.

Case (ii): If $r = 3, 5, 7, \dots$, then $(-1)^{\frac{r(n+1)(n+2)}{2}}$ is not fixed for all $n \geq 1$ as $(-1)^n$ or $(-1)^{2n}$. Thus, Theorem 2.1 and Theorem 2.2 cannot be applied.

Since $S_{\Psi_1} \left(\frac{1}{q} \right)$ is irrational for every integer $q \geq 2$, and $S_{\Psi_1} \left(-\frac{1}{q} \right)$ is irrational when $r = 2, 4, 6, \dots$, so $S_{\Psi_1}(q)$ is irrational at

(i) $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ when $r = 3, 5, 7, \dots$, and

(ii) $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$ when $r = 2, 4, 6, \dots$.

4. Deduction of Irrational Behavior of Bilateral Mock Theta Functions of Order 5, 7, 9, ...

Bilateral mock theta functions of order 5, 7, 9, ... can be obtained by taking $r = 2, 3, 4, \dots$ in (1), (2), ..., (8) respectively. Here, the theorems for irrational behavior of bilateral mock theta functions of order 5, 7, 9, ... have been deduced as particular cases of Theorem 3.1 and Theorem 3.2. For $r = 2, 3$, and 4, respectively, Theorem 3.1 and Theorem 3.2 give the following theorems:

Theorem 4.1 (Theorem 4.1 [11]) *Bilateral mock theta functions of order five $F_0c_2(q)$, $F_1c_2(q^2)$, $\Phi_0c_2(q)$, $\Phi_1c_2(q)$, $f_0c_2(q)$, $f_1c_2(q)$, $\Psi_0c_2(q)$ and $\Psi_1c_2(q)$ take on irrational values at $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$.*

Theorem 4.2 (Theorem 4.1 [13]) *Bilateral mock theta functions $F_0c_3(q)$, $F_1c_3(q^2)$, $\Phi_0c_3(q)$ and $\Phi_1c_3(q)$ take on irrational values at $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$.*

Theorem 4.3 (Theorem 4.2 [13]) *Bilateral mock theta functions $f_0c_3(q)$, $f_1c_3(q)$, $\Psi_0c_3(q)$ and $\Psi_1c_3(q)$ take on irrational values at $q = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$.*

Theorem 4.4 (Theorem 3 [12]) *Bilateral mock theta functions of order nine $F_0c_4(q)$, $F_1c_4(q^2)$, $\Phi_0c_4(q)$, $\Phi_1c_4(q)$, $f_0c_4(q)$, $f_1c_4(q)$, $\Psi_0c_4(q)$ and $\Psi_1c_4(q)$ are irrational at $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$.*

5. Conclusion

In Theorem 3.1 and Theorem 3.2, we discussed the irrational behaviour of bilateral mock theta functions of order $2r + 1$ at $q = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots$, and by taking $r = 2, 3, 4, \dots$ in these theorems, we deduced the theorems for irrational behaviour of bilateral mock theta functions of order 5, 7, 9, ... respectively. By applying this concept, anyone can find the irrational behaviour of other infinite series.

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