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# Ramanujan's Continued Fractions of Order Six and Twelve with Their Interesting Properties

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## Abstract

In the present paper, we derive three continued fractions: $R(q)$, which is of the sixth order, and $S(q)$ and $T(q)$, both of the twelfth order. Each of these arises as a particular case of the general continued fraction mentioned by Ramanujan in his notebook. For these continued fractions, we also establish several theta function identities. In addition, we derive matching coefficients for the continued fraction $S(q)$, and we obtain color partition identities as a result of the theta function identities of $S(q)$ and $T(q)$. Finally, we present 2-, 4-dissections of the continued fraction $S^*(q) = q^{-1}S(q)$.

## Keywords

Ramanujan Theta Function, Continued Fraction

## 1. Introduction

S. Ramanujan (1887-1920) has been perhaps the most widely discussed mathematician of the last century. There has never been a mathematician in history who can evaluate continuing fractions and extend functions in continued fractions like Ramanujan. Ramanujan's work on continued fraction is primarily found in Chapters 12, 14, 16 and the unorganized portion at the end of his Second Notebook [31]. Many beautiful and striking results on continued fractions are found in the Ramanujan 'Lost' Notebook [32]. In this paper, we focus on three interesting continued fraction $R(q)$ of order 6, $S(q)$ and $T(q)$ of order 12 with Ramanujan theta function identities. Here we shall use Ramanujan's general theta-function $f(u, v)$ frequently. The Chapter 16 of his Second Notebook mainly devoted to Ramanujan theta function and its application in $q$-continued fraction. Almost all the Entries of Chapter 16 of his Second Notebook are comprehensively explained and proved by Adiga, Bernt, Bhargava and Watson [1] in their remarkable memoir.

## Ramanujan's Theta-Function

Ramanujan's general theta-function $f(u,v)$ [[10], p.34] is defined as

$$f(u,v)=\sum_{\gamma=-\infty}^{\infty} u^{\gamma(\gamma+1)/2}\ v^{\gamma(\gamma-1)/2}, \qquad |uv|<1. \tag{1}$$

The theta-function $f(u,v)$ can be expressed in the form of the renowned Jacobi Triple Product identity [[10], p.35, Entry 19] as

$$f(u,v)=(-u,-v,uv;uv)_\infty \tag{2}$$

Three noticeable particular instances of $f(u,v)$ are the functions $\phi(q)$, $\psi(q)$ and $f(-q)$ [[10], p.36, Entry 22 (i)-(iii)] presented by

$$\phi(q)=f(q,q)=\sum_{\gamma=-\infty}^{\infty} q^{\gamma^2}=\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}, \tag{3}$$

$$\psi(q)=f(q,q^3)=\sum_{\gamma=0}^{\infty} q^{\frac{\gamma(\gamma+1)}{2}}=\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}, \tag{4}$$

$$f(-q)=f(-q,-q^2)=\sum_{\gamma=-\infty}^{\infty}(-1)^n q^{\frac{\gamma(3\gamma-1)}{2}}=(q;q)_\infty. \tag{5}$$

Ramanujan provided the following definition of the function $\chi(q)$ [[10], p.36, Entry 22 (iv)]

$$\chi(q)=(-q;q^2)_\infty. \tag{6}$$

Just to make expressions easier, we write

$$f_n=f(-q^n)=(q^n;q^n)_\infty. \tag{7}$$

## Ramanujan Continued Fraction

The mostly renowned Rogers-Ramanujan $q$-continued fraction [[1], Entry 38(iii), p. 71] is defined by

$$\frac{f(-q,-q^4)}{f(-q^2,-q^3)}=\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ddots}}}} \qquad |q|<1. \tag{8}$$

Ramanujan recorded on [[31], p.299 ] an interesting continued fractions Ramanujan-Göllnitz-Gordon continued fraction $H(q)$, which is determined by

$$\begin{aligned}H(q)&=q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty}=q^{1/2}\frac{f(-q,-q^7)}{f(-q^3,-q^5)}\\&=\cfrac{q^{1/2}}{1+q+\cfrac{q^2}{1+q^3+\cfrac{q^4}{1+q^5+\ddots}}} \qquad |q|<1.\end{aligned} \tag{9}$$

Ramanujan presented two other identities for $H(q)$ [[31], p.299 ] as

$$\frac{1}{H(q)} - H(q) = \frac{\phi(q^2)}{q^{1/2}\psi(q^4)} \tag{10}$$

$$\frac{1}{H(q)} + H(q) = \frac{\phi(q)}{q^{1/2}\psi(q^4)} \tag{11}$$

Here we derived the continued fractions $R(q)$ of order 6, $S(q)$ and $T(q)$ of order 12, as

$$\begin{aligned} R(q) &= q^{1/4}\frac{f(-q,-q^5)}{f(-q^2,-q^4)} = q^{1/4}\frac{(q,q^5;q^6)_\infty}{(q^2,q^4;q^6)_\infty} \\ &= \cfrac{q^{1/4}(1-q)}{(1-q^{3/2}) + \cfrac{q^{3/2}(1-q^{1/2})(1-q^{5/2})}{(1-q^{3/2})(q^3+1) + \cfrac{q^{3/2}(1-q^{7/2})(1-q^{11/2})}{(1-q^{3/2})(q^6+1)+\ddots}}} \end{aligned} \tag{12}$$

$$\begin{aligned} S(q) &= q\frac{f(-q,-q^{11})}{f(-q^5,-q^7)} = q\frac{(q,q^{11};q^{12})_\infty}{(q^5,q^7;q^{12})_\infty} \\ &= \cfrac{q(1-q)}{(1-q^3) + \cfrac{q^3(1-q^2)(1-q^4)}{(1-q^3)(q^6+1) + \cfrac{q^3(1-q^8)(1-q^{10})}{(1-q^3)(q^{12}+1)+\ddots}}} \end{aligned} \tag{13}$$

$$\begin{aligned} T(q) &= q^{1/2}\frac{f(-q^2,-q^{10})}{f(-q^4,-q^8)} = q^{1/2}\frac{(q^2,q^{10};q^{12})_\infty}{(q^4,q^8;q^{12})_\infty} \\ &= \cfrac{q^{1/2}(1-q^2)}{(1-q^3) + \cfrac{q^3(1-q)(1-q^5)}{(1-q^3)(q^6+1) + \cfrac{q^3(1-q^7)(1-q^{11})}{(1-q^3)(1+q^{12})+\ddots}}} \end{aligned} \tag{14}$$

In this paper, we are concerned with the above continued fractions $R(q)$, $S(q)$ and $T(q)$ and some interesting properties of these continued fractions. In section 3, we derive the continued fraction $R(q)$, $S(q)$ and $T(q)$ as particular cases of the general continued fraction mentioned by Ramanujan [10] in his notebook. We establish certain theta function identities for S(q) and T (q) in section 4, as well as identities linking S(q) and T (q) with the continued fraction R(q). We derive matching coefficients for the continued fraction $S(q)$ in section 5, and color partition identities arising from the continued fraction $S(q)$ and $T(q)$ in section 6. Finally, in section 7, we provide 2-, 4-dissections of the continued fraction $S^*(q) = q^{-1}S(q)$.

## 2. Notations

The following $q$ notations will be followed all over the paper:

**Definition 2.1** *For complex numbers $s$ and $q$ with $\mid q \mid < 1$,*

$$(s;q)_0 = 1,$$

$$(s;q)_k = (1-s)(1-sq)(1-sq^2)\cdots(1-sq^{k-1}) = \prod_{j=1}^{k}(1-sq^{j-1}) \tag{15}$$

$$(s;q)_\infty = (1-s)(1-sq)(1-sq^2)\cdots = \prod_{j=1}^{\infty}(1-sq^{j-1}) \tag{16}$$

*where $q$ is called the base and $s$ the parameter.*

Taking $s = q$ in the above product,

$$(q;q)_\infty = (1-q)(1-q^2)(1-q^3)\cdots = \prod_{k=1}^{\infty}(1-q^k) \tag{17}$$

In brief, we write

$$(s_1;q)_\infty(s_2;q)_\infty(s_3;q)_\infty\cdots(s_m;q)_\infty = (s_1,s_2,s_3,\cdots,s_m;q)_\infty. \tag{18}$$

For a general base $q^l$, $\mid q^l \mid < 1,\ (s;q^l)_0 = 1$ and

$$(s;q^l)_k = (1-s)(1-sq^l)(1-sq^{2l})\cdots(1-sq^{l(k-1)}) = \prod_{j=1}^{k}(1-sq^{l(j-1)}) \tag{19}$$

$$(s;q^l)_\infty = (1-s)(1-sq^l)(1-sq^{2l})\cdots = \prod_{j=1}^{\infty}(1-sq^{l(j-1)}) \tag{20}$$

We also use the Jacobi Triple Product identity (JTP). With base $q$, it is specified as

$$\left(a,\frac{q}{a},q;q\right)_\infty = \prod_{\gamma=1}^{\infty}(1-aq^{\gamma-1})\left(1-\frac{q^\gamma}{a}\right)(1-q^\gamma) = \sum_{\gamma=-\infty}^{\infty}(-1)^\gamma a^\gamma q^{\frac{\gamma(\gamma-1)}{2}} \tag{21}$$

Taking $q \to q^2$, and then $a \to -aq$, we can get another form of Jacobi Triple Product identity with base $q^2$,

$$(-aq,-a^{-1}q,q^2;q^2)_\infty = \prod_{\gamma=1}^{\infty}(1+aq^{2\gamma-1})(1+a^{-1}q^{2\gamma-1})(1-q^{2\gamma}) = \sum_{\gamma=-\infty}^{\infty}a^\gamma q^{\gamma^2} \tag{22}$$

**An easily checked fact** is

$$\sum_{\gamma=-\infty}^{\infty}q^{2\gamma^2+\gamma} = \sum_{\gamma=0}^{\infty}q^{\frac{\gamma^2+\gamma}{2}} = \sum_{\gamma=1}^{\infty}q^{\frac{\gamma^2-\gamma}{2}} \tag{23}$$

Taking $q = 0,-1,1,-2,2,\cdots$ in the bilateral sum $\sum_{\gamma=-\infty}^{\infty}q^{2\gamma^2+\gamma}$, we get

$$\sum_{\gamma=-\infty}^{\infty}q^{2\gamma^2+\gamma} = 1+q+q^3+q^6+q^{10}+q^{15}+q^{21}+\cdots\cdots$$

Also, we have

$$\sum_{\gamma=0}^{\infty}q^{\frac{\gamma^2+\gamma}{2}} = \sum_{\gamma=1}^{\infty}q^{\frac{\gamma^2-\gamma}{2}} = 1+q+q^3+q^6+q^{10}+q^{15}+q^{21}+\cdots$$

Combining the above expressions, we obtain (23).

## 3. The Continued Fractions $R(q)$, $S(q)$ and $T(q)$

In this section, we establish the continued fraction $R(q)$, $S(q)$ and $T(q)$ as particular instances of the following general continued fraction [[1], Entry 12, p.17] mentioned by Ramanujan in his notebook:

Suppose that $\sigma$, $\delta$ and $q$ are complex numbers with $|\sigma q/\delta| < 1$, $|\delta q/\sigma| < 1$ and $|q| < 1$. Then

$$\frac{(\sigma^2 q^3; q^4)_\infty (\delta^2 q^3; q^4)_\infty}{(\sigma^2 q; q^4)_\infty (\delta^2 q; q^4)_\infty} = \cfrac{1}{1 - \sigma\delta + \cfrac{(\sigma - \delta q)(\delta - \sigma q)}{(1 - \sigma\delta)(q^2 + 1) + \cfrac{(\sigma - \delta q^3)(\delta - \sigma q^3)}{(1 - \sigma\delta)(q^4 + 1) + \cdots}}} \tag{24}$$

The above beautiful continued fraction is a $q$-analogue of Entry 25 in Chapter 12 of his Second Notebook [31].

## Proof for $R(q)$:

Replacing $q$ by $q^{3/2}$ in (24), we get

$$\frac{(\sigma^2 q^{9/2}; q^6)_\infty (\delta^2 q^{9/2}; q^6)_\infty}{(\sigma^2 q^{3/2}; q^6)_\infty (\delta^2 q^{3/2}; q^6)_\infty} = \cfrac{1}{1 - \sigma\delta + \cfrac{(\sigma - \delta q^{3/2})(\delta - \sigma q^{3/2})}{(1 - \sigma\delta)(q^3 + 1) + \cfrac{(\sigma - \delta q^{9/2})(\delta - \sigma q^{9/2})}{(1 - \sigma\delta)(q^6 + 1) + \ddots}}} \tag{25}$$

Setting $\sigma = q^{1/4}$, $\delta = q^{5/4}$ in above continued fraction and simplifying, we obtain

$$\frac{(q, q^5; q^6)_\infty}{(q^2, q^4; q^6)_\infty} = \cfrac{1 - q}{(1 - q^{3/2}) + \cfrac{q^{3/2}(1 - q^{5/2})(1 - q^{1/2})}{(1 - q^{3/2})(q^3 + 1) + \cfrac{q^{3/2}(1 - q^{11/2})(1 - q^{7/2})}{(1 - q^{3/2})(q^6 + 1) + \ddots}}}$$

Thus, by using the relation (2) between the Ramanujan theta-function $f(u, v)$ and Jacobi Triple Product Identity, we get

$$\begin{aligned}\frac{f(-q, -q^5)}{f(-q^2, -q^4)} &= \frac{(q, q^5; q^6)_\infty}{(q^2, q^4; q^6)_\infty} \\ &= \cfrac{1 - q}{(1 - q^{3/2}) + \cfrac{q^{3/2}(1 - q^{1/2})(1 - q^{5/2})}{(1 - q^{3/2})(q^3 + 1) + \cfrac{q^{3/2}(1 - q^{7/2})(1 - q^{11/2})}{(1 - q^{3/2})(q^6 + 1) + \ddots}}}\end{aligned} \tag{26}$$

The continued fraction $R(q)$ defined in (12) yields on multiplying $q^{1/4}$ to the above expression.

## Proof for $S(q)$:

Again, replacing $q$ by $q^3$ in (24), we get

$$\frac{(\sigma^2 q^9;q^{12})_\infty(\delta^2 q^9;q^{12})_\infty}{(\sigma^2 q^3;q^{12})_\infty(\delta^2 q^3;q^{12})_\infty} = \cfrac{1}{1-\sigma\delta+\cfrac{(\sigma-\delta q^3)(\delta-\sigma q^3)}{(1-\sigma\delta)(q^6+1)+\cfrac{(\sigma-\delta q^9)(\delta-\sigma q^9)}{(1-\sigma\delta)(q^{12}+1)+\ddots}}} \tag{27}$$

Setting $\sigma = q,\ \delta = q^2$ in (27) and simplifying, we obtain

$$\frac{(q,q^{11};q^{12})_\infty}{(q^5,q^7;q^{12})_\infty} = \cfrac{1-q}{(1-q^3)+\cfrac{q^3(1-q^4)(1-q^2)}{(1-q^3)(q^6+1)+\cfrac{q^3(1-q^{10})(1-q^8)}{(1-q^3)(q^{12}+1)+\ddots}}}$$

By using the relation (2), we get

$$\begin{aligned}\frac{f(-q,-q^{11})}{f(-q^5,-q^7)} &= \frac{(q,q^{11};q^{12})_\infty}{(q^5,q^7;q^{12})_\infty} \\ &= \cfrac{1-q}{(1-q^3)+\cfrac{q^3(1-q^2)(1-q^4)}{(1-q^3)(q^6+1)+\cfrac{q^3(1-q^8)(1-q^{10})}{(1-q^3)(q^{12}+1)+\ddots}}}\end{aligned} \tag{28}$$

The expression $S(q)$ defined in (13) turns by multiplying $q$ to the above continued fraction.

## Proof for $T(q)$:

Setting $\sigma = q^{1/2}$, $\delta = q^{5/2}$ in (24) and simplifying, we obtain

$$\frac{(q^2,q^{10};q^{12})_\infty}{(q^4,q^8;q^{12})_\infty} = \cfrac{1-q^2}{(1-q^3)+\cfrac{q^3(1-q^5)(1-q)}{(1-q^3)(q^6+1)+\cfrac{q^3(1-q^{11})(1-q^7)}{(1-q^3)(1+q^{12})+\ddots}}}$$

Hence by using the relation (2), the above continued fraction becomes

$$\begin{aligned}\frac{f(-q^2,-q^{10})}{f(-q^4,-q^8)} &= \frac{(q^2,q^{10};q^{12})_\infty}{(q^4,q^8;q^{12})_\infty} \\ &= \cfrac{1-q^2}{(1-q^3)+\cfrac{q^3(1-q)(1-q^5)}{(1-q^3)(q^6+1)+\cfrac{q^3(1-q^7)(1-q^{11})}{(1-q^3)(1+q^{12})+\ddots}}}\end{aligned} \tag{29}$$

On multiplying $q^{1/2}$ to the above expression, it turns to the continued fraction $T(q)$ defined in (14).

## 4. Identities for $S(q)$ and $T(q)$ in terms of Theta-functions

Here, we establish a few identities for the continued fractions $S(q)$ and $T(q)$ involving functions $\phi(q),\ \psi(q),\ \chi(-q)$, and the fraction $R(q)$:

**Theorem 4.1**

$$\text{(i)}\quad \frac{1}{S(q)} - S(q) = \frac{f(-q^2)\ \phi(q^3)}{q\ \psi(q^3)\ \chi(-q)\ \psi(q^6)},$$

$$\text{(ii)}\quad \frac{1}{S(q)} - S(q) = \frac{\phi(q^3)}{q^{3/4}\ \psi(q^6)}\frac{1}{R(q)},$$

$$\text{(iii)}\quad \frac{1}{T(q)} - T(q) = \frac{\psi(q^3)\ \chi(-q)\ \phi(q^3)}{q^{1/2}\ f(-q^2)\ \psi(q^6)},$$

$$\text{(iv)}\quad \frac{1}{T(q)} - T(q) = \frac{\phi(q^3)}{q^{3/4}\ \psi(q^6)}R(q),$$

$$\text{(v)}\quad \frac{1}{S(q)} + S(q) = \frac{f(q^2,q^4)\ \phi(-q^3)}{q\ f(-q,-q^5)\ \psi(q^6)} = \frac{f(q^2,q^4)\ \phi(-q^3)}{q\ \psi(q^3)\ \chi(-q)\ \psi(q^6)},$$

$$\text{(vi)}\quad \frac{1}{T(q)} + T(q) = \frac{f(q,q^5)\ \phi(-q^3)}{q^{1/2}\ f(-q^2,-q^4)\ \psi(q^6)} = \frac{\psi(-q^3)\ \chi(q)\ \phi(-q^3)}{q^{1/2}\ f(-q^2)\ \psi(q^6)},$$

$$\text{(vii)}\quad \left(\frac{1}{S(q)} - S(q)\right)\left(\frac{1}{T(q)} - T(q)\right) = \frac{\phi^2(q^3)}{q^{3/2}\ \psi^2(q^6)}$$

$$\text{(viii)}\quad \frac{(T^{-1}(q) - T(q))}{(S^{-1}(q) - S(q))} = R^2(q).$$

**Proof:**

**(i)** From (13), taking $S(q) = q\frac{f(-q,-q^{11})}{f(-q^5,-q^7)}$, we obtain

$$\frac{1}{\sqrt{S(q)}} - \sqrt{S(q)} = \frac{f(-q^5,-q^7) - qf(-q,-q^{11})}{\sqrt{qf(-q,-q^{11})f(-q^5,-q^7)}}. \tag{30}$$

From [[10], Entry 30 (ii) and (iii), p.46], we have

$$f(u,v) = f(u^3v, uv^3) + uf\left(\frac{v}{u}, u^5v^3\right) \tag{31}$$

Taking $u = -q$ and $v = q^2$ in above identity, we get

$$f(-q,q^2) = f(-q^5,-q^7) - qf(-q,-q^{11}) \tag{32}$$

Employing (32) in (30), we find that

$$\frac{1}{\sqrt{S(q)}} - \sqrt{S(q)} = \frac{f(-q,q^2)}{\sqrt{qf(-q,-q^{11})f(-q^5,-q^7)}} \tag{33}$$

In a similar way, we conclude that

$$\frac{1}{\sqrt{S(q)}} + \sqrt{S(q)} = \frac{f(-q^5,-q^7) + qf(-q,-q^{11})}{\sqrt{qf(-q,-q^{11})f(-q^5,-q^7)}}. \tag{34}$$

Taking $u = q$ and $v = -q^2$ in (31), we have

$$f(q,-q^2) = f(-q^5,-q^7) + qf(-q,-q^{11}) \tag{35}$$

Employing (35) in (34), we get

$$\frac{1}{\sqrt{S(q)}} + \sqrt{S(q)} = \frac{f(q,-q^2)}{\sqrt{qf(-q,-q^{11})f(-q^5,-q^7)}} \tag{36}$$

Multiplying (33) and (36), we get

$$\frac{1}{S(q)} - S(q) = \frac{f(-q,q^2)f(q,-q^2)}{qf(-q,-q^{11})f(-q^5,-q^7)} \tag{37}$$

From [[10], Entry 30 (i), (iv), page 46], we have

$$f(u,uv^2)f(v,u^2v) = f(u,v)\psi(uv) \tag{38}$$

and

$$f(u,v)f(-u,-v) = f(-u^2,-v^2)\phi(-uv) \tag{39}$$

Taking $u = -q$ and $v = -q^5$ in (38), we get

$$f(-q,-q^{11})f(-q^5,-q^7) = f(-q,-q^5)\psi(q^6) \tag{40}$$

Taking $u = -q$ and $v = q^2$ in (39), we get

$$f(-q,q^2)f(q,-q^2) = f(-q^2,-q^4)\phi(q^3) \tag{41}$$

Substituting (40) and (41) in (37), we get

$$\frac{1}{S(q)} - S(q) = \frac{f(-q^2,-q^4)\phi(q^3)}{qf(-q,-q^5)\psi(q^6)} \tag{42}$$

From [[1], Ex. (v), page 49], we have

$$\begin{aligned} f(q,q^5) &= \psi(-q^3)\chi(q) \\ f(-q,-q^5) &= \psi(q^3)\chi(-q) \end{aligned} \tag{43}$$

Replacing $q$ by $q^2$ in (5), we get

$$f(-q^2) = f(-q^2,-q^4) \tag{44}$$

By using (43) and (44), the identity (42) reduces to Theorem 4.1 (i).

**(ii)** From (12), we have

$$\frac{f(-q^2,-q^4)}{f(-q,-q^5)} = \frac{q^{1/4}}{R(q)} \tag{45}$$

Using (45), the identity (42) reduces to the Theorem 4.1 (ii).

**(iii)** From (14), $T(q) = q^{1/2}\frac{f(-q^2,-q^{10})}{f(-q^4,-q^8)}$, so

$$\frac{1}{\sqrt{T(q)}} - \sqrt{T(q)} = \frac{f(-q^4,-q^8) - q^{1/2}f(-q^2,-q^{10})}{\sqrt{q^{1/2}f(-q^2,-q^{10})f(-q^4,-q^8)}}. \tag{46}$$

Taking $u = -q^{1/2}$ and $v = q^{5/2}$ in (31), we get

$$f(-q^{1/2},q^{5/2}) = f(-q^4,-q^8) - q^{1/2}f(-q^2,-q^{10}) \tag{47}$$

Employing the above identity in (46), we find that

$$\frac{1}{\sqrt{T(q)}} - \sqrt{T(q)} = \frac{f(-q^{1/2},q^{5/2})}{\sqrt{q^{1/2}f(-q^2,-q^{10})f(-q^4,-q^8)}} \tag{48}$$

Furthermore, we infer from (14) that

$$\frac{1}{\sqrt{T(q)}} + \sqrt{T(q)} = \frac{f(-q^4,-q^8) + q^{1/2}f(-q^2,-q^{10})}{\sqrt{q^{1/2}f(-q^2,-q^{10})f(-q^4,-q^8)}}. \tag{49}$$

Taking $u = q^{1/2}$ and $v = -q^{5/2}$ in (31), we have

$$f(q^{1/2},-q^{5/2}) = f(-q^4,-q^8) + q^{1/2}f(-q^2,-q^{10}) \tag{50}$$

Employing (50) in (49), we get

$$\frac{1}{\sqrt{T(q)}} + \sqrt{T(q)} = \frac{f(q^{1/2},-q^{5/2})}{\sqrt{q^{1/2}f(-q^2,-q^{10})f(-q^4,-q^8)}} \tag{51}$$

Multiplying (48) and (51), we get

$$\frac{1}{T(q)} - T(q) = \frac{f(-q^{1/2},q^{5/2})f(q^{1/2},-q^{5/2})}{q^{1/2}f(-q^2,-q^{10})f(-q^4,-q^8)} \tag{52}$$

Taking $u = -q^2$ and $v = -q^4$ in (38), we get

$$f(-q^2,-q^{10})f(-q^4,-q^8) = f(-q^2,-q^4)\psi(q^6) \tag{53}$$

Taking $u = -q^{1/2}$ and $v = q^{5/2}$ in (39), we get

$$f(-q^{1/2},q^{5/2})f(q^{1/2},-q^{5/2}) = f(-q,-q^5)\phi(q^3) \tag{54}$$

Substituting (53) and (54) in (52), we get

$$\frac{1}{T(q)} - T(q) = \frac{f(-q,-q^5)\ \phi(q^3)}{q^{1/2}f(-q^2,-q^4)\ \psi(q^6)} \tag{55}$$

By using (43) and (44), the identity (55) reduces to Theorem 4.1 (iii).

**(iv)** From (12), we have

$$\frac{f(-q,-q^5)}{f(-q^2,-q^4)} = \frac{R(q)}{q^{1/4}} \tag{56}$$

Using (56), the identity (55) reduces to Theorem 4.1 (iv).

**(v)** On squaring (36), we have

$$\frac{1}{S(q)} + S(q) = \frac{f^2(q,-q^2)}{qf(-q,-q^{11})f(-q^5,-q^7)} - 2 \tag{57}$$

From [[10], Entry 30 (v),(vi), page 46], we have

$$f^2(u,v) = f(u^2,v^2)\ \phi(uv) + 2u\ f\left(\frac{v}{u}, u^3v\right)\ \psi(u^2v^2) \tag{58}$$

Taking $u = q$ and $v = -q^2$ in (58), the identity becomes

$$f^2(q,-q^2) = f(q^2,q^4)\phi(-q^3) + 2qf(-q,-q^5)\psi(q^6) \tag{59}$$

Substituting (59) and (40) in (57), we obtain

$$\frac{1}{S(q)} + S(q) = \frac{f(q^2,q^4)\ \phi(-q^3)}{q\ f(-q,-q^5)\ \psi(q^6)} \tag{60}$$

By using (43), the identity (60) reduces to Theorem 4.1 (v).

**(vi)** On squaring (51), we have

$$\frac{1}{T(q)} + T(q) = \frac{f^2(q^{1/2},-q^{5/2})}{q^{1/2}f(-q^2,-q^{10})f(-q^4,-q^8)} - 2 \tag{61}$$

Taking $u = q^{1/2}$ and $v = -q^{5/2}$ in (58), the identity becomes

$$f^2(q^{1/2},-q^{5/2}) = f(q,q^5)\phi(-q^3) + 2q^{1/2}f(-q^2,-q^4)\psi(q^6) \tag{62}$$

Substituting (53) and (62) in (61), we obtain

$$\frac{1}{T(q)} + T(q) = \frac{f(q,q^5)\ \phi(-q^3)}{q^{1/2}\ f(-q^2,-q^4)\ \psi(q^6)} \tag{63}$$

By using (43) and (44), the identity (63) reduces to Theorem 4.1 (vi).

**(vii)** When we multiply Theorem 4.1(ii) with Theorem 4.1(iv), we obtain Theorem 4.1(vii).

**(viii)** Dividing Theorem 4.1(iv) by Theorem 4.1(ii) results in Theorem 4.1(viii).

## 5. Matching Coefficients For $S(q)$

This section presents the results for the matching coefficients, derived from the identities in Theorem 4.1, for $S(q)$ with its reciprocal. Baruah and Das [6] present a number of matching coefficient values for the reciprocals and series expansion for specific $q$-products. The definition of the matching coefficients, as given in reference [6], is as follows:

**Definition 5.1** *Two power series $\sum_{\gamma=0}^{\infty} U_\gamma q^\gamma$ and $\sum_{\gamma=0}^{\infty} V_\gamma q^\gamma$ are said to have matching coefficients provided for some positive integers $i, j$ and $k$, and non negative integers $l$ and $m$, $U(i\gamma + l) = \pm kV(j\gamma + m)$, for all $\gamma \geq 0$.*

To prove matching coefficients results for the continued fraction $S(q)$ with its reciprocal, we shall use the following two lemmas:

**Lemma 5.2**

$$(i) \quad \phi(q^3) = \frac{f_6^5}{f_3^2 f_{12}^2} \qquad (ii) \quad \psi(q^3) = \frac{f_6^2}{f_3} \qquad (iii) \quad \psi(q^6) = \frac{f_{12}^2}{f_6}$$
$$(iv) \quad f(-q^2) = f_2 \qquad (v) \quad \chi(-q) = \frac{f_1}{f_2}.$$

**Proof (i)** Taking $q$ to $q^3$ in (3) and using (7), we get

$$\phi(q^3) = \frac{(q^6; q^6)_\infty^5}{(q^3; q^3)_\infty^2 (q^{12}; q^{12})_\infty^2} = \frac{f_6^5}{f_3^2 f_{12}^2}.$$

**(ii)** Taking $q$ to $q^3$ in (4) and using (7), we get

$$\psi(q^3) = \frac{(q^6; q^6)_\infty^2}{(q^3; q^3)_\infty} = \frac{f_6^2}{f_3}.$$

**(iii)** Similarly, by taking $q$ to $q^6$ in (4) and using (7), we get

$$\psi(q^6) = \frac{(q^{12}; q^{12})_\infty^2}{(q^6; q^6)_\infty} = \frac{f_{12}^2}{f_6}.$$

**(iv)** From (7), we have

$$f(-q^2) = (q^2; q^2)_\infty = f_2.$$

**(v)** Taking $q$ to $-q$ in (6) and using (7), we get

$$\chi(-q) = (q; q^2)_\infty = \frac{(q; q)_\infty}{(q^2; q^2)_\infty} = \frac{f_1}{f_2}.$$

**Lemma 5.3** *From ([7],[25],[26]), we have*

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q \frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}.$$

**Theorem 5.4**

$$\begin{aligned} \text{If} \quad & qS(q) = q^2 \frac{(q, q^{11}; q^{12})_\infty}{(q^5, q^7; q^{12})_\infty} = \sum_{\gamma=0}^{\infty} c_\gamma q^\gamma \\ \text{and} \quad & q \frac{1}{S(q)} = \frac{(q^5, q^7; q^{12})_\infty}{(q, q^{11}; q^{12})_\infty} = \sum_{\gamma=0}^{\infty} c'_\gamma q^\gamma, \\ \text{then} \quad & c_{4\gamma+3} = c'_{4\gamma+3}. \end{aligned} \tag{64}$$

**Proof:** As, we have

$$S(q) = \frac{q(q, q^{11}; q^{12})_\infty}{(q^5, q^7; q^{12})_\infty},$$

so by using this fraction in Theorem 4.1(i), we get

$$\frac{(q^5, q^7; q^{12})_\infty}{q(q, q^{11}; q^{12})_\infty} - \frac{q(q, q^{11}; q^{12})_\infty}{(q^5, q^7; q^{12})_\infty} = \frac{f(-q^2)\ \phi(q^3)}{q\ \psi(q^3)\ \chi(-q)\ \psi(q^6)} \tag{65}$$

Multiplying both sides by $q$, and using Lemma 5.2, the above identity becomes

$$\frac{(q^5,q^7;q^{12})_\infty}{(q,q^{11};q^{12})_\infty}-\frac{q^2(q,q^{11};q^{12})_\infty}{(q^5,q^7;q^{12})_\infty}=\frac{f_2^2f_6^4}{f_1f_3f_{12}^4}$$

By using Lemma 5.3, the above identity reduces to

$$\frac{(q^5,q^7;q^{12})_\infty}{(q,q^{11};q^{12})_\infty}-\frac{q^2(q,q^{11};q^{12})_\infty}{(q^5,q^7;q^{12})_\infty}=\frac{f_2^2f_6^4}{f_{12}^4}\left[\frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2}+q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}\right] \tag{66}$$

Using (64) in the above expression, we have

$$\sum_{\gamma=0}^{\infty}c'_\gamma q^\gamma-\sum_{\gamma=0}^{\infty}c_\gamma q^\gamma=\frac{f_8^2f_{12}}{f_4f_{24}^2}+q\frac{f_6^2f_4^5f_{24}^2}{f_2^2f_8^2f_{12}^5}$$

Extracting the terms involving $q^{2\gamma+1}$, we get

$$\sum_{\gamma=0}^{\infty}c'_{2\gamma+1}q^{2\gamma+1}-\sum_{\gamma=0}^{\infty}c_{2\gamma+1}q^{2\gamma+1}=q\frac{f_6^2f_4^5f_{24}^2}{f_2^2f_8^2f_{12}^5}$$

Dividing by $q$ and then replacing $q^2$ by $q$, we obtain

$$\sum_{\gamma=0}^{\infty}c'_{2\gamma+1}q^{\gamma}-\sum_{\gamma=0}^{\infty}c_{2\gamma+1}q^{\gamma}=\frac{f_3^2f_2^5f_{12}^2}{f_1^2f_4^2f_6^5}$$

Again extracting the terms involving $q^{2\gamma+1}$ in above expression, dividing by $q$ and then replacing $q^2$ by $q$, we obtain

$$\sum_{\gamma=0}^{\infty}c'_{4\gamma+3}q^{\gamma}=\sum_{\gamma=0}^{\infty}c_{4\gamma+3}q^{\gamma}\qquad\text{[no term on } q^{2\gamma+1}\text{ on right side]}$$

Equating the coefficient of $q^\gamma$ on both sides, we get

$$c'_{4\gamma+3}=c_{4\gamma+3}.$$

## 6. Color Partition Identities For $S(q)$ and $T(q)$

In this section, the color partition identities have been derived from the theta function identities of $S(q)$ and $T(q)$. Before deriving the color partition identities, we need the following definitions:

**Definition 6.1** *We can write a positive integer $v$ as a sum of positive integers as,*

$$v=v_1+v_2+\cdots+v_k$$

*with $v_1\geq v_2\geq\cdots\geq v_k$ (and $k\geq 1$), then the number of ways that the positive integer $v$ can be written as the above sum is called the partition function and denoted by $p(v)$. We call the integers $v_j$'s as the parts of the partition. It's not necessary for the parts to be distinct, and two partitions are same if they differ only in the order of their parts. Consider $p(0)$=1.*

For examples,

$$3 = 3 = 2+1 = 1+1+1, \qquad p(3) = 3.$$
$$4 = 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1, \qquad p(4) = 5.$$

Euler[19] discovered the generating function for the partition function which is given as

$$\sum_{v=0}^{\infty} p(v)q^v = \prod_{v=1}^{\infty}(1-q^v)^{-1} = \frac{1}{(q;q)_\infty}.$$

**Definition 6.2** *Colored partitions are the partitions of a positive integer into parts with colors. In $k$-color partition of a positive integer $v$, each part of a partition appears in $k$ distinct colors. We write $P_k(v)$ to represent the number of partition of $v$ with $k$ different colors for each part.*

**Example:** The number of 2-color partition of 3 are

$$3_g,\ 3_b,\ 2_g+1_g,\ 2_g+1_b,\ 2_b+1_g,\ 2_b+1_b,\ 1_g+1_g+1_g,\ 1_g+1_g+1_b,\ 1_g+1_b+1_b,\ 1_b+1_b+1_b,$$

where the suffixes $g$ and $b$ denote the colors green and blue respectively. Thus, $P_2(3) = 10$.

The generating function for $P_k(v)$ is given by

$$\sum_{v=0}^{\infty} P_k(v)q^v = \frac{1}{(q;q)_\infty^k} \tag{67}$$

For the positive integers $i, j$ and $k$, the number of partitions of $v$ having each part $k$ colors with parts congruent to $i \equiv \mod j$ is given by the generating function,

$$\frac{1}{(q^i;q^j)_\infty^k}$$

Suppose that if $r$ and $s$ are positive integers, then the generating function of the number of partitions of positive integer having each part $k$ colors with parts congruent to $r \equiv \mod j$ or $s \equiv \mod j$ is

$$\frac{1}{(q^r;q^j)_\infty^k (q^s;q^j)_\infty^l} = \frac{1}{(q^r,q^s;q^j)_\infty^k}$$

For simplicity, we employ the notation

$$(q^{i\pm};q^j)_\infty = (q^i, q^{j-i};q^j)_\infty \tag{68}$$

where $i$ and $j$ are positive integers and $i < j$.

**Theorem 6.3** *Let the number of partitions of $v$ into parts congruent to $\pm1, \pm2, \pm5$ or $\pm6$ (mod 12) such that the parts congruent to $\pm2$ and $\pm6$ (mod 12) with 2 colors be denoted by $A(v)$, and the number of partitions of $v$ into parts congruent to $\pm1, \pm4, \pm5$ or $\pm6$ (mod 12) such that the parts congruent to $\pm4$ and $\pm6$ (mod 12) with 2 colors denoted by $B(v)$. If the number of partitions of $v$ into parts congruent to $\pm2, \pm3$ or $\pm4$ (mod 12) with 2 colors is denoted by $C(v)$, then for any integer $v \geq 1$,*

$$A(v) - B(v-1) - C(v) = 0. \tag{69}$$

**Proof:** From (14), (3), (4) and (12), we have

$$T(q) = q^{1/2}\frac{(q^2,q^{10};q^{12})_\infty}{(q^4,q^8;q^{12})_\infty}, \quad \phi(q) = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2},$$
$$\psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}, \quad R(q) = q^{1/4}\frac{(q,q^5;q^6)_\infty}{(q^2,q^4;q^6)_\infty}$$

Using above expressions in the Theorem 4.1 (iv), we get

$$\frac{(q^4,q^8;q^{12})_\infty}{q^{1/2}(q^2,q^{10};q^{12})_\infty} - q^{1/2}\frac{(q^2,q^{10};q^{12})_\infty}{(q^4,q^8;q^{12})_\infty}$$
$$= \frac{(q^6;q^6)_\infty^6}{q^{1/2}(q^3;q^3)_\infty^2(q^{12};q^{12})_\infty^4}\cdot\frac{(q;q^6)_\infty(q^5;q^6)_\infty}{(q^2;q^6)_\infty(q^4;q^6)_\infty}$$

Multiplying both sides by $q^{1/2}$ in the above expression, we get

$$\frac{(q^4,q^8;q^{12})_\infty}{(q^2,q^{10};q^{12})_\infty} - \frac{q(q^2,q^{10};q^{12})_\infty}{(q^4,q^8;q^{12})_\infty} = \frac{(q^6;q^6)_\infty^6}{(q^3;q^3)_\infty^2(q^{12};q^{12})_\infty^4}\cdot\frac{(q;q^6)_\infty(q^5;q^6)_\infty}{(q^2;q^6)_\infty(q^4;q^6)_\infty}$$

Now using the Notation (68) in the above, we obtain

$$\frac{(q^{4\pm};q^{12})_\infty}{(q^{2\pm};q^{12})_\infty} - \frac{q(q^{2\pm};q^{12})_\infty}{(q^{4\pm};q^{12})_\infty} = \frac{(q^6;q^6)_\infty^6}{(q^3;q^3)_\infty^2(q^{12};q^{12})_\infty^4}\cdot\frac{(q;q^6)_\infty(q^5;q^6)_\infty}{(q^2;q^6)_\infty(q^4;q^6)_\infty}$$

Changing each base to base $q^{12}$, as

$$(q;q^6)_\infty = (q,q^7;q^{12})_\infty\ , \qquad (q^2;q^6)_\infty = (q^2,q^8;q^{12})_\infty\ ,$$
$$(q^4;q^6)_\infty = (q^4,q^{10};q^{12})_\infty\ , \qquad (q^5;q^6)_\infty = (q^5,q^{11};q^{12})_\infty\ ,$$
$$(q^6;q^6)_\infty = (q^6,q^{12};q^{12})_\infty\ , \qquad (q^3;q^3)_\infty = (q^3,q^6,q^9,q^{12};q^{12})_\infty\ ,$$

we get

$$\frac{(q^{4\pm};q^{12})_\infty}{(q^{2\pm};q^{12})_\infty} - \frac{q(q^{2\pm};q^{12})_\infty}{(q^{4\pm};q^{12})_\infty} - \frac{(q^{1\pm,5\pm};q^{12})_\infty(q^{6\pm};q^{12})_\infty^2}{(q^{2\pm,4\pm};q^{12})_\infty(q^{3\pm};q^{12})_\infty^2} = 0 \tag{70}$$

Dividing the above by $(q^{1\pm,2\pm,4\pm,5\pm};q^{12})_\infty\ (q^{6\pm};q^{12})_\infty^2$, we obtain

$$\frac{1}{(q^{1\pm,5\pm};q^{12})_\infty(q^{2\pm,6\pm};q^{12})_\infty^2} - \frac{q}{(q^{1\pm,5\pm};q^{12})_\infty(q^{4\pm,6\pm};q^{12})_\infty^2} - \frac{1}{(q^{2\pm,3\pm,4\pm};q^{12})_\infty^2} = 0$$

The aforementioned quotients show the generating functions for $A(v)$, $B(v)$ and $C(v)$ respectively. Hence, the above expression is written as

$$\sum_{v=0}^{\infty} A(v)q^v - q\sum_{v=0}^{\infty} B(v)q^v - \sum_{v=0}^{\infty} C(v)q^v = 0 \tag{71}$$

where we take $A(0) = B(0) = C(0) = 1$.

Comparing the coefficients of $q^v$ on both sides of (71), the identity (69) yields.

**Theorem 6.4** *Let $\alpha(v)$ be the number of partitions of $v$ into parts congruent to $\pm1, \pm2, \pm4$ or $\pm6$ (mod 12) such that the parts congruent to $\pm1$ and $\pm6$ (mod 12) with 2 colors, and $\beta(v)$ be the number of partitions of $v$ into parts congruent to $\pm2, \pm4, \pm5$ or $\pm6$ (mod 12) such that the parts congruent to $\pm5$ and $\pm6$ (mod 12) with 2 colors. If the number of partitions of $v$ into parts congruent to $\pm1, \pm3$ or $\pm5$ (mod 12) with 2 colors is denoted by $\delta(v)$, then for any integer $v \geq 1$,*

$$\alpha(v) - \beta(v-2) - \delta(v) = 0. \tag{72}$$

**Proof:** Next, from (13), (3), (4) and (12), we have

$$S(q) = q\frac{(q, q^{11}; q^{12})_\infty}{(q^5, q^7; q^{12})_\infty}, \quad \phi(q) = \frac{(q^2; q^2)^5_\infty}{(q; q)^2_\infty (q^4; q^4)^2_\infty},$$
$$\psi(q) = \frac{(q^2; q^2)^2_\infty}{(q; q)_\infty}, \quad R(q) = q^{1/4}\frac{(q, q^5; q^6)_\infty}{(q^2, q^4; q^6)_\infty}$$

Employing the above expressions in Theorem 4.1 (ii), we get

$$\frac{(q^5, q^7; q^{12})_\infty}{q(q, q^{11}; q^{12})_\infty} - q\frac{(q, q^{11}; q^{12})_\infty}{(q^5, q^7; q^{12})_\infty}$$
$$= \frac{(q^6; q^6)^6_\infty}{q(q^3; q^3)^2_\infty (q^{12}; q^{12})^4_\infty} \cdot \frac{(q^2; q^6)_\infty (q^4; q^6)_\infty}{(q; q^6)_\infty (q^5; q^6)_\infty}$$

Multiplying both sides by $q$ in the above expression, we get

$$\frac{(q^5, q^7; q^{12})_\infty}{(q, q^{11}; q^{12})_\infty} - \frac{q^2(q, q^{11}; q^{12})_\infty}{(q^5, q^7; q^{12})_\infty} = \frac{(q^6; q^6)^6_\infty}{(q^3; q^3)^2_\infty (q^{12}; q^{12})^4_\infty} \cdot \frac{(q^2; q^6)_\infty (q^4; q^6)_\infty}{(q; q^6)_\infty (q^5; q^6)_\infty}$$

Now using the Notation (68) in the above, we obtain

$$\frac{(q^{5\pm}; q^{12})_\infty}{(q^{1\pm}; q^{12})_\infty} - \frac{q^2(q^{1\pm}; q^{12})_\infty}{(q^{5\pm}; q^{12})_\infty} = \frac{(q^6; q^6)^6_\infty}{(q^3; q^3)^2_\infty (q^{12}; q^{12})^4_\infty} \cdot \frac{(q^2; q^6)_\infty (q^4; q^6)_\infty}{(q; q^6)_\infty (q^5; q^6)_\infty}$$

Changing each base to base $q^{12}$ and simplifying, we get

$$\frac{(q^{5\pm}; q^{12})_\infty}{(q^{1\pm}; q^{12})_\infty} - \frac{q^2(q^{1\pm}; q^{12})_\infty}{(q^{5\pm}; q^{12})_\infty} - \frac{(q^{2\pm,4\pm}; q^{12})_\infty (q^{6\pm}; q^{12})^2_\infty}{(q^{1\pm,5\pm}; q^{12})_\infty (q^{3\pm}; q^{12})^2_\infty} = 0 \tag{73}$$

Dividing the above equation by $(q^{1\pm,2\pm,4\pm,5\pm}; q^{12})_\infty \ (q^{6\pm}; q^{12})^2_\infty$, we obtain

$$\frac{1}{(q^{2\pm,4\pm}; q^{12})_\infty (q^{1\pm,6\pm}; q^{12})^2_\infty} - \frac{q^2}{(q^{2\pm,4\pm}; q^{12})_\infty (q^{5\pm,6\pm}; q^{12})^2_\infty} - \frac{1}{(q^{1\pm,3\pm,5\pm}; q^{12})^2_\infty} = 0$$

The generating functions for the aforementioned quotients are $\alpha(v)$, $\beta(v)$ and $\delta(v)$ respectively. Hence, the above expression is equivalent to

$$\sum_{v=0}^{\infty} \alpha(v) q^v - q^2 \sum_{v=0}^{\infty} \beta(v) q^v - \sum_{v=0}^{\infty} \delta(v) q^v = 0 \tag{74}$$

where we set $\alpha(0) = \beta(0) = \delta(0) = 1$. On comparing the coefficients of $q^v$ on both sides of (74), the identity (72) yields.

## 7. Dissection formula for $S^*(q)$

**Definition 7.1** *For an integer $\sigma > 1$, to express a given series as a sum of $\sigma$ sums is called $\sigma$-dissection.*

As an example, $\phi(q)$ has the following 2-dissection [From [25]]:

$$\phi(q) = \phi(q^4) + 2q\psi(q^8).$$

Here we give 2- and 4-dissection of the continued fraction $S^*(q) = q^{-1}S(q)$.

**Theorem 7.2** *If*

$$S^*(q) = q^{-1}S(q) = \frac{f(-q,-q^{11})}{f(-q^5,-q^7)} = \sum_{\gamma=0}^{\infty} d_\gamma q^\gamma, \tag{75}$$

*then 2-dissection of* $S^*(q)$ *are*

$$\sum_{\gamma=0}^{\infty} d_{2\gamma} q^\gamma = \frac{f(-q^4)f(-q^3,-q^9)}{\phi(-q^6)f(-q^5,-q^7)}. \tag{76}$$

$$\sum_{\gamma=0}^{\infty} d_{2\gamma+1} q^\gamma = -\frac{\psi(q^6)\chi(-q^2)f(-q^3,-q^9)}{\phi(-q^6)f(-q^5,-q^7)} \tag{77}$$

**Proof:** From (75), we have

$$\sum_{\gamma=0}^{\infty} d_\gamma q^\gamma = \frac{f(-q,-q^{11})}{f(-q^5,-q^7)} \cdot \frac{f(q^5,q^7)}{f(q^5,q^7)} \tag{78}$$

From [[10],p. 45, Entry 29], we have

$$f(u,v)f(r,s) = f(ur,vs)f(us,vr) + uf(v/r,ur^2s)f(v/s,urs^2);\ \text{for}\quad uv = rs \tag{79}$$

Setting $u = -q,\ v = -q^{11},\ r = q^5\ \&\ s = q^7$ in the above formula, we get

$$f(-q,-q^{11})f(q^5,q^7) = f(-q^6,-q^{18})f(-q^8,-q^{16}) - qf(-q^6,-q^{18})f(-q^4,-q^{20}) \tag{80}$$

Setting $u = q^5,\ v = q^7$ in (39), we get

$$f(q^5,q^7)\ f(-q^5,-q^7) = f(-q^{10},-q^{14})\ \phi(-q^{12}) \tag{81}$$

Employing (80) and (81) in (78), we get

$$\sum_{\gamma=0}^{\infty} d_\gamma q^\gamma = \frac{f(-q^6,-q^{18})f(-q^8,-q^{16}) - qf(-q^6,-q^{18})f(-q^4,-q^{20})}{f(-q^{10},-q^{14})\ \phi(-q^{12})} \tag{82}$$

Replace $q$ by $-q$ in above, we obtain

$$\sum_{\gamma=0}^{\infty} d_\gamma (-q)^\gamma = \frac{f(-q^6,-q^{18})f(-q^8,-q^{16}) + qf(-q^6,-q^{18})f(-q^4,-q^{20})}{f(-q^{10},-q^{14})\ \phi(-q^{12})} \tag{83}$$

Since,

$$\sum_{\gamma=0}^{\infty} d_\gamma q^\gamma + \sum_{\gamma=0}^{\infty} d_\gamma (-q)^\gamma = 2\sum_{\gamma=0}^{\infty} d_{2\gamma} q^{2\gamma}, \tag{84}$$

So adding (82) and (83), we get

$$\sum_{\gamma=0}^{\infty} d_{2\gamma} q^{2\gamma} = \frac{f(-q^6,-q^{18})\ f(-q^8,-q^{16})}{f(-q^{10},-q^{14})\ \phi(-q^{12})}. \tag{85}$$

Also,

$$\sum_{\gamma=0}^{\infty} d_\gamma q^\gamma - \sum_{\gamma=0}^{\infty} d_\gamma (-q)^\gamma = 2\sum_{\gamma=0}^{\infty} d_{2\gamma+1} q^{2\gamma+1} \tag{86}$$

Subtracting (83) from (82), we get

$$\sum_{\gamma=0}^{\infty} d_{2\gamma+1}q^{2\gamma+1} = -q\frac{f(-q^6,-q^{18})\ f(-q^4,-q^{20})}{f(-q^{10},-q^{14})\ \phi(-q^{12})} \tag{87}$$

Dividing both sides by $q$, we get

$$\sum_{\gamma=0}^{\infty} d_{2\gamma+1}q^{2\gamma} = -\ \frac{f(-q^6,-q^{18})\ f(-q^4,-q^{20})}{f(-q^{10},-q^{14})\ \phi(-q^{12})} \tag{88}$$

Replacing $q^2$ by $q$ in (85) and (88) and using (5) and (43), we obtain the 2-dissections of $S^*(q)$ as given by (76) and (77) respectively.

**Theorem 7.3** *The 4-dissections of $S^*(q)$ are*

$$\sum_{\gamma=0}^{\infty} d_{4\gamma}q^{\gamma} = \frac{f(-q^2)f(-q^4,-q^8)}{\phi(-q^3)\phi(-q^6)} \tag{89}$$

$$\sum_{\gamma=0}^{\infty} d_{4\gamma+1}q^{\gamma} = \ \frac{\psi(q^3)\chi(-q)f(-q^4,-q^8)}{\phi(-q^3)\phi(-q^6)} \tag{90}$$

$$\sum_{\gamma=0}^{\infty} d_{4\gamma+2}q^{\gamma} = -q\frac{f(-q^2)f(-q^2,-q^{10})f(-q,-q^{11})}{\phi(-q^3)\phi(-q^6)f(-q^5,-q^7)} \tag{91}$$

$$\sum_{\gamma=0}^{\infty} d_{4\gamma+3}q^{\gamma} = -q\frac{\psi(q^3)\chi(-q)f(-q^2,-q^{10})f(-q,-q^{11})}{\phi(-q^3)\phi(-q^6)f(-q^5,-q^7)}. \tag{92}$$

**Proof:** From (76), we have

$$\sum_{\gamma=0}^{\infty} d_{2\gamma}q^{\gamma} = \frac{f(-q^4)f(-q^3,-q^9)}{\phi(-q^6)f(-q^5,-q^7)} \cdot \frac{f(q^5,q^7)}{f(q^5,q^7)} \tag{93}$$

Setting $u=-q^3,\ v=-q^9,\ r=q^5\ \ \&\ \ s=q^7$ in (79), we have

$$f(-q^3,-q^9)f(q^5,q^7) = f(-q^8,-q^{16})f(-q^{10},-q^{14}) - q^3 f(-q^4,-q^{20})f(-q^2,-q^{22}) \tag{94}$$

Using (94) and (81), the sum (93) becomes

$$\sum_{\gamma=0}^{\infty} d_{2\gamma}q^{\gamma} = \frac{f(-q^4)[f(-q^8,-q^{16})f(-q^{10},-q^{14}) - q^3 f(-q^4,-q^{20})f(-q^2,-q^{22})]}{\phi(-q^6)\ f(-q^{10},-q^{14})\ \phi(-q^{12})} \tag{95}$$

Replacing $q$ by $-q$ in above, we obtain

$$\sum_{\gamma=0}^{\infty} d_{2\gamma}(-q)^{\gamma} = \frac{f(-q^4)[f(-q^8,-q^{16})f(-q^{10},-q^{14}) + q^3 f(-q^4,-q^{20})f(-q^2,-q^{22})]}{\phi(-q^6)\ f(-q^{10},-q^{14})\ \phi(-q^{12})} \tag{96}$$

Adding (95) and (96) and using (84), we get

$$\sum_{\gamma=0}^{\infty} d_{4\gamma}q^{2\gamma} = \frac{f(-q^4)f(-q^8,-q^{16})}{\phi(-q^6)\phi(-q^{12})} \tag{97}$$

Subtracting (96) from (95) and using (86), we get

$$\sum_{\gamma=0}^{\infty} d_{4\gamma+2}q^{2\gamma+1} = -q^3 \frac{f(-q^4)f(-q^4,-q^{20})f(-q^2,-q^{22})}{\phi(-q^6)f(-q^{10},-q^{14})\phi(-q^{12})} \tag{98}$$

Dividing both side by $q$, we obtain

$$\sum_{\gamma=0}^{\infty} d_{4\gamma+2}q^{2\gamma} = -q^2 \frac{f(-q^4)f(-q^4,-q^{20})f(-q^2,-q^{22})}{\phi(-q^6)f(-q^{10},-q^{14})\phi(-q^{12})} \tag{99}$$

Replacing $q^2$ by $q$ in (97) and (99), we obtain (89) and (91) respectively.

Next, from (77), we have

$$\sum_{\gamma=0}^{\infty} d_{2\gamma+1}q^{\gamma} = -\frac{f(-q^3,-q^9)\psi(q^6)\chi(-q^2)}{\phi(-q^6)f(-q^5,-q^7)} \cdot \frac{f(q^5,q^7)}{f(q^5,q^7)} \tag{100}$$

Employing (94) and (81) in (100), we get

$$\sum_{\gamma=0}^{\infty} d_{2\gamma+1}q^{\gamma} = \frac{\psi(q^6)\chi(-q^2)[f(-q^8,-q^{16})f(-q^{10},-q^{14}) - q^3 f(-q^4,-q^{20})f(-q^2,-q^{22})]}{\phi(-q^6)f(-q^{10},-q^{14})\phi(-q^{12})} \tag{101}$$

Replace $q$ by $-q$ in above, we get

$$\sum_{\gamma=0}^{\infty} d_{2\gamma+1}(-q)^{\gamma} = \frac{\psi(q^6)\chi(-q^2)[f(-q^8,-q^{16})f(-q^{10},-q^{14}) + q^3 f(-q^4,-q^{20})f(-q^2,-q^{22})]}{\phi(-q^6)f(-q^{10},-q^{14})\phi(-q^{12})} \tag{102}$$

Adding (101) and (102) and using (84), we get

$$\sum_{\gamma=0}^{\infty} d_{4\gamma+1}q^{2\gamma} = \frac{\psi(q^6)\ \chi(-q^2)\ f(-q^8,-q^{16})}{\phi(-q^6)\ \phi(-q^{12})} \tag{103}$$

Subtracting (102) from (101) and using (86), we get

$$\sum_{\gamma=0}^{\infty} d_{4\gamma+3}q^{2\gamma+1} = -q^3 \frac{\psi(q^6)\ \chi(-q^2)\ f(-q^4,-q^{20})\ f(-q^2,-q^{22})}{\phi(-q^6)\ f(-q^{10},-q^{14})\ \phi(-q^{12})} \tag{104}$$

Dividing both sides by $q$, we obtain

$$\sum_{\gamma=0}^{\infty} d_{4\gamma+3}q^{2\gamma} = -q^2 \frac{\psi(q^6)\ \chi(-q^2)\ f(-q^4,-q^{20})\ f(-q^2,-q^{22})}{\phi(-q^6)\ f(-q^{10},-q^{14})\ \phi(-q^{12})} \tag{105}$$

Replacing $q^2$ by $q$ in (103) and (105), we obtain (90) and (92) respectively.

## 8. Conclusion

Here we derived the continued fraction $R(q)$, $S(q)$ and $T(q)$ as particular cases of the Ramanujan general continued fraction, and some identities for $S(q)$ and $T(q)$ in terms of theta-functions. We also presented matching coefficients results arising from $S(q)$, the color partition identities arising from $S(q)$ and $T(q)$, and the 2-, 4-dissections of the continued fraction $S^*(q) = q^{-1}S(q)$. At last we conclude that one can get different continued fractions from Ramanujan general continued fraction and reveal many interesting properties related to theta function identities, continued fractions, partition function etc.

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